

Modeling cognitive systems with Category Theory

Towards rigor in cognitive sciences

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Abstract

Traditionally, mathematical formalisms in cognitive sciences have been confined to toy model world descriptions. (5)

In the absence of a theory written in mathematical terms, the separation between the different disciplines that form the cognitive sciences will be progressively more acute and an understanding between them unattainable.

This paper claims for a shift towards the formal sciences in cognitive sciences.

In particular, category theory is proposed, through the applications and example given below, as a fitting tool for the building of an unified theory of cognition.

Category theory is presented as a valid foundational framework for modeling in cognitive sciences. Brain structure modeling, perception models or semantic models of neural networks are sketched under this *categorical* outlook.

1 Introduction

A mathematical explanation of how the brain is structured and how it achieves cognitive functions like perception, conceptualization or learning is seen for cognitive scientists, especially for those of humanistic background, as an extreme reductionism that obviates essential human capabilities like agency, intention, intuition, emotions or feelings.

The so called "hard sciences" have achieved mathematical certainty about the world, and although it has been always the case that formal systems fall short of capturing reality completely, this is not because these systems (mind included) are impossible to formally explain, rather it is us, as modelers who seem to have limited (perceptual) access to reality, so that we, unfortunately, usually only get partially valid descriptions.

We claim that, the object of study of the cognitive sciences, how the brain works, is still in a pre-scientific stage. The progress from an immature science to a mature one must pass through the construction of appropriate mechanisms with explanatory power.

The main objective of this paper is to set the agenda of category theory as that appropriate mechanism that provides the theoretical framework in order to build cognitive models in mathematical terms.

In (6), E.C. Zeeman holds that a mathematical explanation of how the brain works has to rely on the concept of isomorphism.

First develop a piece of mathematics X that describes the permanent structure (memory) and the working (thinking) of the mind, and another piece of mathematics Y that describes the permanent structure (anatomy) and working (electromechanical) of the brain; then, from hypothesis based on experimental evidence, prove an isomorphism $X \cong Y$

Of course, we must be cautious with *experimental evidence* about brain studies,

mentioned in a paper of 1962. We must be cautious and question, update and discard the outdated empirical data used in the theory.

The approach and the emphasizes given to the concepts structure and morphism are however, totally relevant. This paper defends the idea that structure and morphism are core concepts that we need to study and put into a formal context in order to get an unified theory of cognition.

There are two main issues we must take from this statement.

First, if Category theory is the mathematical theory that studies the structure, we can not prescind such theory when we try to model the structure and the function of the brain.

Second, morphism in Category theory, generalizes the notion of mapping. Establishing the relation of representation between two systems involves the establishment of a mapping -i.e: morphism, ideally an isomorphism.

2 The impasse of modelling the brain. Two obstacles and one way out

Next, we introduce two major difficulties in modelling cognitive systems as complicated as the brain. The former is the excess of empirical data from fMRI studies and the later is the view that the brain configures metric maps that reflect objects and processes from the world out there.

1) Undeniably the brain is a very malleable system, but the theories and models of the brain do not have to be that malleable. The proliferation of fMRI experiments are not necessarily producing a better understanding of brain morphology and functioning.

2) One of the biggest challenges in science today is to decipher the map of the brain. In Crick's words, "there is little hope to understand how the brain works until we have a map of the neural wiring in the mammalian brain." (7)

The way out of the two above problems is to approach the study of the structure of the brain with a systemic and mathematical based approach. Mathematics provides provable knowledge about the real world, and this is due to the fact that mathematical structures deal better than anything else with the structure of the world. That is to say, there is a morphism or structure preserving mapping, between the mathematical structure that models the world and the world itself.

The main idea is to translate the concept of algebraic structure into brain studies. Around the notion of structure as a repeated pattern, critical concepts such as neural function or neural semantics will achieve formal expressions.

The next section outlines a very brief introduction of category theory. Next, in

sections 4, 5 and 6. we present three different domains where category theory is being used.

3 Category Theory

Category theory is a theory of mathematical structure based upon the notion of arrow. An arrow, also called morphism, represents a relationship between two objects in a category. However counterintuitive it may be, in category theory maps between objects rather than objects themselves take the primary role.(1) (2)

A category is composed of the following data: 1) Objects $A, B, C \dots$; 2) Morphisms or arrows $f, g, h \dots$ between objects; 3) For each $f : A \rightarrow B$ there is a domain $\text{dom}(f)=A$ and a codomain $\text{cod}(f)=B$; 4) For each morphism, $f : A \rightarrow B$ and $g : B \rightarrow C$, a composition rule, $(f \circ g) : A \rightarrow C$. Note that $\text{cod}(f) = \text{dom}(g)$; 5) The composition is associative, $((h \circ g) \circ f)(a) = h(g(f(a))) = (h \circ (g \circ f))(a)$; 6) every object A has an identity map $1_a : A \rightarrow A$ given by $1_a(a) = a$

A category is *anything* that satisfies this definition, the objects can be sets, groups, monoids, vector spaces...or neurons in the hippocampus. The arrows used to be functions, but is not always the case. Hence, a category is nothing more than an algebra of arrows equipped with the composition operator.

4 Applications of Category Theory in Cognitive Systems modeling

M. Healy (9), describes a mathematical semantic model for neural networks based upon Category theory. The core of the model proposed by Healy consists of using categorical terms like colimit, limit or functor in order to “bring mathematical rigor to the understanding of knowledge representations in neural networks”.

A network architecture A , formed of a set of neurons, together with an array of the connection weight values w of that set, is modeled as the category $N_{A,w}$. An object of $N_{A,w}$ is defined by a pair (p_i, w) , where set $p_i = 1, 2, \dots, n_k$ is the nodes of A and w represents the set of output values for p_i connections. A morphism $m : (p_i, w) \rightarrow (p_j, w')$ of $N_{A,w}$ is defined by a set of connection paths (synapsis) between the set of nodes (neurons) and their weight states, (p_i, w) , and the nodes (neurons) and their weight states, (p_j, w') .

The category **Concept** is defined as a colimit of other simpler concepts which are defined as categories in the model. The functors transport the invariant structure across the category **Concept** and the category $N_{A,w}$, $M : \mathbf{Concept} \rightarrow N_{A,w}$

The main idea is that learning can be modeled as a transition between categories. A *Functor* is used to model the structure-preserving associations between

categories. On the other hand, *Colimits* express the learning of more complex concepts through the re-use of simpler concepts already represented in the connection weight memory of a neural network.

5 Category Theory for Perception

In (8) Z. Arzi-Gonczarowski deploys a basic category theory tool for perception modeling purposes. A perception is a 3-tuple $\langle E, I, \rho \rangle$ such that E and I are finite, disjoint sets and ρ is the arrow $\rho : E \times I \rightarrow \{t, f, u\}$. The set E are elements of the external world and I are mental concepts or internal connotations of the external world. Therefore the predicate $\rho : E \times I \rightarrow \{t, f, u\}$ is 3-valued. A mental concept i , can be a true, false or unknown, connotation of an external object.

Let E be an environment and $P_1 = \langle I_1, \rho_1 \rangle$ and $P_2 = \langle I_2, \rho_2 \rangle$ two perceptions over E .

The mapping $h : P_1 \rightarrow P_2$ is a perception morphism (p-morphism) iff h is a mapping between the connotations I_1 and I_2 and definite truth values (t,f) are preserved by the p-morphism. Note that p-morphisms are the categorical morphisms of the category based on the collection of all perceptions with the same environment E .

6 Category Theory for Consciousness

The mathematician A.C. Ehresmann and the physicist J.P. Vanbremersch, have spent 20 years working together on an mathematical model based upon category theory models for studying living organisms. The model Memory Evolutive System(3) provides a formal unified model for the investigation of the mind, translating ideas of neuroscientists into a mathematical language.

The fundamental question of *how higher mental processes arise from the functioning of the brain?* is approached by the formation of increasingly complex objects. In this vein, neurons (**Neur**), category of neurons and mental objects (**ImO** that stands for Image of O) are models of the brain at different hierarchical levels.

Thus, the category **Neur** is composed of neurons and models the physical structure of the brain and its elementary neural dynamics. The binding of a pattern P of neurons in a category of level 1 (**Neur**), becomes the mental image ImO of an object O . Thus, ImO is a *cat-neuron* of level 1.

Progressively, the construction of a cat-neuron of higher level, 2 and so on, is established by the mental image of an object C formed by the juxtaposition of several objects O_i that the animal can already recognize.

7 Conclusions

The data without a theory are mere noise. We are living times of major technical advances in measurement and of detailed explanation at the cellular and molecular level. Nevertheless, the global picture of brain functioning is still missing. The time has come to set the agenda for a “hard cognitive science”. To that end, the authors propose to translate into mathematical terms, some key concepts like perception or mental objects that until now have been used loosely, and are lacking mathematical structure. The applications shown in this paper, depict the suitability of Category theory as a language for complex system modeling, and as sophisticated toolkit for mental theories.

References

- [1] S. MAC LANE, *Categories for the Working Mathematician.*, Springer, 1971.
- [2] W. W. LAWVERE AND S. H. SCHANUEL, *Conceptual Mathematics: A First Introduction to Categories*, Cambridge University Press, 1995.
- [3] A. C. EHRESMANN AND J. P. VANBREMEERSCH, *Memory evolutive systems: hierarchy, emergence, cognition*, Elsevier Science, 2007.
- [4] M. BODEN , *Mind as machine*, Oxford University Press, 2006.
- [5] H. DREYFUS , *What Computers Can't Do: The Limits of Artificial Intelligence*, Harper & Row, New York, 1972.
- [6] E.C. ZEEMAN, *The Topology of the Brain and Visual Perception*, Topology of 3-Manifolds, M. K. Fort, Jr., Ed., Prentice-Hall (1962) 240–256.
- [7] J. LICHTMAN, *Making connections: By turning neurons technicolour*, Jeff Lichtman exposed the brain's wiring, Nature (2009) January 28.
- [8] Z. ARZI ARZI-GONCZAROWSKI AND D. J. LEHMANN, *From Environments to Representations: A Mathematical Theory of Artificial Perceptions*, Artificial Intelligence (1998) 187–247.
- [9] M. HEALY, *Category Theory Applied to Neural Modeling and Graphical Representations*, Proceedings of IJCNN 2000, IEEE (2000).

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