The mathematical structures of the brain. Category Theory in neural science

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Abstract

Brain science is still in adolescence [doi:10.4249/scholarpedia.8040]. Indeed, the mechanisms that govern critical features of brain function, for example learning or memory are still under debate. While a lot is known about the properties and functions of neurons, the concepts for translating the data from the micro level to the macro level and vice-verse, are still uncertain.
Contrary to physics, that is built on formal theories embedded in universal laws, rendering particulars and details unnecessary; in biology, particular cases are still relevant.
We claim for a shift towards a science of complexity able to deal with complex structures like human brains and their functioning.

The main objective of this work is the introduction of a theoretical foundation for neuroscience based on mathematical structures to enable the transition of cognitive science from its current heterogeneous state into a rigorous, systematic and convergent scientific endeavor.
The sciences of the complex, using mathematical methods, provide provable knowledge about the real world, this is due to the fact that mathematics principally deals with structure.
Concepts from Category Theory like **colimit** or **functor** are proposed as universal tools that should be applied to the different levels of organization of the brain. Brain structure modeling will be sketched under this categorical outlook. The time is ripe for the new systemic and mathematical brain sciences.

A new framework for place and grid cells

How does the mind represent physical space?. This is a question that has kept philosophers busy for centuries.
In 1975, O'Keefe and Nadel, discovered the place cells in the hippocampus of the rat.

Place cells code the Cartesian position
The place cells code explicit (no contextual) locations in the environment.
The region in which a place cell fires the most is called place field.
Granted that place cells are correlated to space, the question that arises is: Are the place cells the only neurons correlated to space?

Grid cells

Grid cells, likewise place cells, are place-modulated neurons.

The firing location of a grid cell is multiple, contrary to the place cells which are mono field.

The multiple firing location of a grid cell is indeed a grid with a most striking property, it is an array of equilateral triangles.

Fig. 2 Grid map of a rat running on a linear track after 10 min [5].

Fig.1 The picture shows the 36 place fields out of each of 36 rat hippocampal CA1 place cells. Thus, as it is shown in the figure the position of the rat is sustained in the firing of these cells. The place fields are cone shapes, this means that that the firing rates increase irrespective of the direction from which the rat arrives.

Place cells and Grid cells, similarities and differences

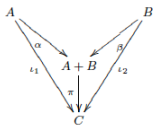
	Brain area	Type of map	Activation
Place cells	Hippocampus	static	need calibration
Grid cells	mEC	dynamic	active instantaneously in any novel environment

The old Kant was correct, the concept of distance or metric actually is embedded in our neuronal tissue as the grid cells show.

- The 70's was the decade of the **place cells**, neurons that discharge when the rat is in a particular position.
- In the 80's **head direction** cells, neurons that discharge significantly whenever the rat's head changes direction
- Since 2005 we have been in the **grid cell** era.
- Yet the underlying nature of the cognitive map remains elusive

A new framework for place and grid cells based on category theory

Definition 4. A coproduct of two objects A and B is an an object $A+B$ together with the arrows $A \xrightarrow{\iota_1} A+B$ and $B \xrightarrow{\iota_2} A+B$, such that for any object C and the pair of arrows $A \xrightarrow{\alpha} C$, $B \xrightarrow{\beta} C$, it exists an unique morphism π that makes the diagram commutes.



The colimit is the mathematical structure that allow us to encode the emergence of place field and the relationship between grid fields.

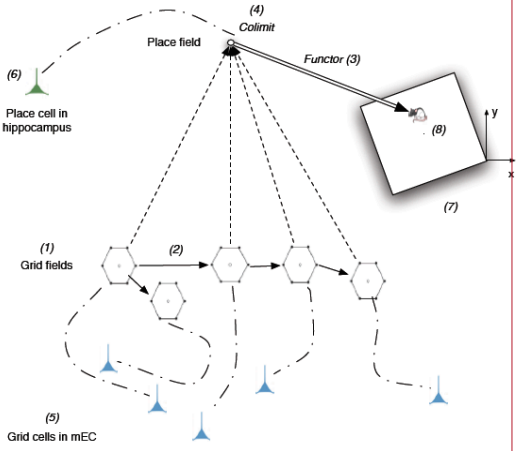


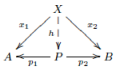
Fig. 6 The figure depicts a colimit where (4) acts as the place field of a place cell(6) in the hippocampus. The colimit is produced by several grid fields (one grid field(1) is produced by one grid cell(5)).

When the coordinated activity of a group grid cells produce a place cell, this is a colimit and it is unique. But given a place cell, its place field cannot be uniquely identified determined by a group of grid cells, several grid fields are possible for that place field.

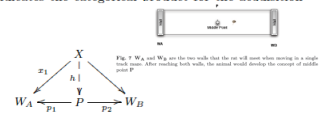
Towards a theory of declarative memory based on category theory

$$Memory = \begin{cases} Declarative\ memory \begin{cases} Episodic \\ Semantic \end{cases} \\ Procedural\ memory \end{cases}$$

Definition 7. In a category $\mathbf{C}$, a product of two objects A and B, is another object P equipped with two morphisms, $P \xrightarrow{p_1} A$ and $P \xrightarrow{q_1} B$, such that for any pair of morphisms, $X \xrightarrow{f} A$ and $X \xrightarrow{g} B$ there is an unique morphism h making the following diagram commutes.



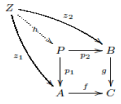
Note that the broken arrow h means that is unique, and the morphisms p_1, p_2 are usually called projection morphisms. The main characteristic of a product is that the constituents are retrievable via the projection morphism. The following diagram indicates the categorical product for the acquisition of the middle point.



For our purpose, the categorical product given by the object P and the morphisms p_1, p_2 is a statement about a cognitive behavior of the rat, whereas X and x_1, x_2 is a constraint on what constitutes a valid product, rather than a specific claim about cognition.

Definition 8. In a category $\mathbf{C}$, a pullback of two morphisms with common codomain $A \xrightarrow{f} C \xleftarrow{g} B$ is an object P together with a pair of morphisms $P \xrightarrow{p_1} A$ and $P \xrightarrow{p_2} B$ that form a commutative diagram $f \circ p_1 = g \circ p_2$.

Moreover, the morphisms are universal among such squares because for any pair of morphisms $Z \xrightarrow{z_1} A$ and $Z \xrightarrow{z_2} B$ such that $f \circ z_1 = g \circ z_2$, there is an unique morphism h such that the following diagram commutes



Conclusions

The goal is not to obtain an all-encompassing theory (or at least not at first!)

Complex systems needs of multiple views. We need to enrich the theoretical context in neuroscience

Creative act: not just to apply existing theories but the ability to create new theories that structurally match biological organisation and dynamics

To equate theoretical neuroscience with computations is due to the wrong assumption that the principles are already known.
The main task in to discover those fundamental Principles or Laws.